Quantum Optics I (Class 2)

Glauber: $\boxed{\hat{\vec{E}}^{(+)}(\vec{r},t) = i\sum_k \left(\frac{1}{2}\hbar\omega_k\right)^{1/2} \hat{a}_k \vec{u}_k(\vec{r})\, e^{-i\omega_k t}}$ δ-normalized

States of the Electromagnetic Field:

Question: what are eigenstates of $\hat{E}^{(+)}(\vec{r},t)$?

i.e eigenstates of $\hat{a}_k$? i.e $a_k|\ \rangle = \alpha_k|\ \rangle$? eigenstate of field?

COHERENT STATES.

$a|\alpha\rangle = \alpha|\alpha\rangle$ thus $\langle\alpha|\hat{a}|\alpha\rangle = \alpha$

$\langle\alpha|\hat{a}^+|\alpha\rangle = \alpha^*$

Thus: $\langle\alpha|a^+a|\alpha\rangle = \alpha\cdot\alpha^* = |\alpha|^2$ i.e mean photon number of a coherent state.

Coherent states satisfy: $|\alpha\rangle = e^{-|\alpha|^2/2}\sum \frac{\alpha^n|n\rangle}{\sqrt{n!}}$

Thus: $p(n) = |\langle\alpha|n\rangle|^2 = \frac{|\alpha|^{2n} e^{-|\alpha|^2}}{n!}$ POISSON DISTRIBUTION

The coherent state is a displaced vacuum state:

$D(\alpha) \equiv \exp(\alpha\hat{a}^+ - \alpha^* a)$

operator theorem in QM: $e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{-[\hat{A},\hat{B}]/2}$ BAKER HAUSENDORF for $[[\hat{A},\hat{B}],\hat{A}] = 0$

$D(\alpha) = e^{-|\alpha|^2/2} e^{\alpha\hat{a}^+} e^{-\alpha^* a}$

and $\boxed{|\alpha\rangle = D(\alpha)|0\rangle}$

Proof: $D^+(\alpha)\hat{a}|\alpha\rangle = D^+(\alpha)\hat{a}D(\alpha)|0\rangle = (\hat{a}+\alpha)|0\rangle = \alpha|0\rangle$

using the property: $D^+(\alpha) = D^{-1}(\alpha) = D(-\alpha)$ | ie $D^+(\alpha)D(\alpha) = \mathbb{1}$

Definition of D according to Glauber: $D^+(\alpha)\hat{a}D(\alpha) \stackrel{!}{=} \hat{a}+\alpha$ (H.W) (*)

$D^+(\alpha)\hat{a}^+D(\alpha) \stackrel{!}{=} \hat{a}^+ + \alpha^*$

Coherent States are not a Basis:

Homework $|\langle\alpha|\beta\rangle|^2 = \exp\left(-\frac{1}{2}(|\alpha|^2+|\beta|^2)+\alpha\beta^*\right)$ $\left(\hat{=} \left(\sum_n \frac{\beta^{*n}}{\sqrt{n!}}\langle n|\right)\left(\sum_m \frac{\alpha^m}{\sqrt{m!}}|m\rangle\right) e^{-|\alpha|^2/2} e^{-|\beta|^2/2}\right.$ $\langle n|m\rangle = \delta_{nm}$.

$= \exp(|\alpha-\beta|^2)$ $\hat{=} \langle 0|D^+(\beta)D(\alpha)|0\rangle$

Note that since a is non Hermetian $(a \neq \hat{a}^+)$ the eigenvalue is COMPLEX.

The coherent states form an overcomplete basis:

$\frac{1}{\pi}\int|\alpha\rangle\langle\alpha|\,d^2\alpha = \mathbb{1}$

Proof of $D^{-1}(\alpha)\,a\,D(\alpha) = D^+(\alpha)\,a\,D(\alpha) = a+\alpha$:

$D^{-1}(\alpha)\,a\,D(\alpha) = e^{\alpha^* a} e^{-\alpha a^+}\, a\, e^{\alpha a^+} e^{-\alpha^* a}$ $\underset{\sim}{\hat{a}^+}\ \underset{\sim}{\hat{a}^+}\ \underset{\sim}{\hat{a}^*}$

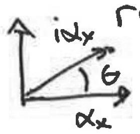
and $e^{-\alpha\hat{A}}\hat{B}e^{\alpha\hat{A}} = \hat{B} - \alpha[\hat{A},\hat{B}] + \frac{\alpha^2}{2!}\underbrace{[\hat{A},[\hat{A},\hat{B}]]}_{=0} + \dots$

Proof of overcomplete basis:

Assume a general state $|A\rangle = \sum_u A_u |u\rangle$

and $\alpha = \alpha_x + i\alpha_y$ (complex)

$re^{i\theta} = \alpha$



$$\frac{1}{\pi}\int d^2\alpha\left(|\alpha\rangle\langle\alpha|\right)|A\rangle = \int \frac{1}{\pi}\sum_u d^2\alpha A_u\, |\alpha\rangle \underbrace{\langle\alpha|u\rangle}_{\to \text{ last page } P(u)}$$

substitute polar coordinates: $\quad \alpha = re^{i\theta}$

$\quad d^2\alpha = \partial\alpha_x\,\partial\alpha_y = r\,dr\,d\theta$

Thus:

$$\frac{1}{\pi}\sum_{m,u}\int_0^\infty\int_0^{2\pi} dr\,d\theta . r . e^{-r^2} r^{u+m}\frac{e^{i\theta(m-u)}}{\sqrt{m!u!}}|m\rangle$$

using: $\langle\alpha|u\rangle = \langle re^{i\theta}|u\rangle = \left(\frac{r^u e^{-iu\theta}}{\sqrt{u!}}\cdot\right).e^{-r^2/2}$

and $|\alpha\rangle = \sum_m \frac{\alpha^m}{\sqrt{m!}}.|m\rangle\, e^{-|\alpha|^2/2} \hat{=} \sum_m \frac{r^m e^{+im\theta}\, e^{-|r|^2/2}}{\sqrt{m!}}|m\rangle$

NOTE: $\langle\alpha| = \sum_u \frac{(\alpha^*)^u}{\sqrt{u!}}\langle u|\, e^{-|\alpha|^2/2}$

Thus:

$$\underbrace{\int_0^{2\pi}\frac{e^{i(m-u)\theta}}{\pi}d\theta}_{2\delta_{mu}} = 2\delta_{m,n}$$

Therefore: $= 2\sum_u\int_0^\infty A_u e^{-r^2}\frac{r^{2u+1}}{(u!)}|u\rangle\, dr$

using $\int_0^\infty e^{-r^2} r^{2u+1} = \int_0^\infty e^{-r^2+\ln(2u+1)} \underset{\uparrow}{=} u!/2$

(without proof)

Thus: combining:

$= \left(\sum_u A_u|u\rangle\right)$ ie our original state (completes proof)

$$\Rightarrow \boxed{\frac{1}{\pi}\int d^2\alpha\, |\alpha\rangle\langle\alpha| = \frac{1}{\pi}\int d\alpha_x d\alpha_y\, |\alpha\rangle\langle\alpha| = \mathbb{1}}$$

(ie) a Basis.

States of the Electromagnetic Field:

COHERENT STATES.

$a|\alpha\rangle = \alpha|\alpha\rangle$ thus $\boxed{\begin{array}{l}\langle\alpha|\hat{a}|\alpha\rangle = \alpha \\ \langle\alpha|\hat{a}^{+}|\alpha\rangle = \alpha^{*}\end{array}}$

Thus: $\langle\alpha|a^{+}a|\alpha\rangle = \alpha\cdot\alpha^{*} = |\alpha|^2$ i.e mean photon number of a coherent state.

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Thus: $p(n) = |\langle\alpha|n\rangle|^2 = \frac{|\alpha|^{2n}e^{-|\alpha|^2}}{n!}$ POISSON DISTRIBUTION

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$D(\alpha) = e^{-|\alpha|^2/2}e^{\alpha\hat{a}^{+}}e^{-\alpha^{*}a}$

and $\boxed{|\alpha\rangle = D(\alpha)|0\rangle}$

Proof: $D^{+}(\alpha)\hat{a}|\alpha\rangle = D^{+}(\alpha)\hat{a}D(\alpha)|0\rangle = (\hat{a}+\alpha)|0\rangle = \alpha|0\rangle$

using the property: $D^{+}(\alpha) = D^{-1}(\alpha) = D(-\alpha)$

$D^{+}(\alpha)\hat{a}D(\alpha) = \hat{a}+\alpha$ (H.W)

Coherent States are not a Basis:

$$|\langle\alpha|\beta\rangle|^2 = \exp(-\tfrac{1}{2}(|\alpha|^2 + |\beta|^2) + \alpha\beta^{*})$$
$$= \exp(|\alpha-\beta|^2)$$

Note that since $\hat{a}$ is non Hermitian $(\hat{a}\neq\hat{a}^{+})$ the eigenvalue is COMPLEX.

The coherent states form an overcomplete basis:

$$\frac{1}{\pi}\int |\alpha\rangle\langle\alpha|\,d^2\alpha = \mathbb{1}$$

Quantum Optics I   Lecture Note 2

Ch 10 Gardiner Stedy. Methods

Coherent state properties

Evolution of the coherent state $|\alpha,t\rangle$

$$\hat{H}|\alpha,t\rangle = -i\hbar\partial_t|\alpha,t\rangle$$

Expand $|\alpha,t\rangle = \sum_n \langle n|\alpha,t\rangle |n\rangle$ (i.e. $\sum_n |n\rangle\langle n| = \mathbb{1}$)

$$i\hbar\partial_t|\alpha,t\rangle = -i\hbar\sum_n |n\rangle\,\partial_t\langle\alpha,t|n\rangle = \hat{H}\sum_n |n\rangle\langle n|\alpha,t\rangle$$

$$= \sum_n \underbrace{(\hbar\omega)(n+\tfrac{1}{2})}_{E_n}|n\rangle\langle n|\alpha,t\rangle$$

$\Leftrightarrow$ $-i\hbar\,\partial_t\langle\alpha,t|n\rangle = \hbar\omega(n+\tfrac{1}{2})\langle n|\alpha,t\rangle$

$$\Leftrightarrow \langle\alpha,t|n\rangle = \overbrace{e^{-i\omega n t + \hbar\omega/2}}^{e^{-iE_n/\hbar\, t}}\langle\alpha,t=0|n\rangle$$

Therefore e.g. for a Fock state

$$|1,t\rangle = |1\rangle e^{-i\omega t} \quad \text{or} \quad |3,t\rangle = |3\rangle e^{-3i\omega t}$$

NOTE $\Rightarrow$ This property we will explore in the context of QUANTUM METROLOGY since it shows that $|n\rangle$ state exhibits a de Broglie wavelength of λ/n where λ is the vacuum wavelength ($\rightarrow$ homework)

returning to the ~~coherent~~ coherent state this property implies:

$$|\alpha,t\rangle = e^{-|\alpha|^2/2}\sum_n \frac{\alpha^n}{\sqrt{n!}}\underbrace{|n\rangle e^{-i\omega t\cdot n}}_{|n,t\rangle} = e^{-|\alpha|^2/2}\sum_n \frac{(\alpha e^{i\omega t})^n}{\sqrt{n!}}|n\rangle$$

Thus the time dependence of $|\alpha\rangle$ is simply given by the eigenvalue $\alpha(t) = \alpha(0)e^{-i\omega t}$ (despite the individual's Fock state time dependence of $\propto i\omega\cdot \underline{\underline{n}}\cdot t$).

Basic Properties of Coherent States

$$a|\alpha\rangle = \alpha|\alpha\rangle \qquad \langle\alpha|a^+ = \alpha^*\langle\alpha|$$

operator ordering matters. One distinguishes
NORMALLY ORDERED OPERATORS; eg:

$$\underset{\substack{\uparrow \\ \text{normal order}}}{:}\hat{a}^+\hat{a}\,\hat{a}^+: \;\hat{=}\; \hat{a}^+\hat{a}^+\hat{a}$$

i.e (creation operators to the right)

note that:

$$\langle\alpha|\hat{a}^+\hat{a}\,\hat{a}^+|\beta\rangle = \langle\alpha|\hat{a}^+(\hat{a}^+\hat{a}+1)|\beta\rangle$$

$$= \langle\alpha|\underbrace{\hat{a}^+\hat{a}^+a}_{\text{normally ordered}} + \hat{a}^+|\beta\rangle \quad \text{using } [\hat{a},\hat{a}^+]=1$$

$$= (\alpha^*\alpha^*\beta + \alpha^*)\langle\alpha|\beta\rangle$$

Normal ordering is in particular important in e.g optical photodetection $(\langle\hat{I}\rangle \propto \langle\hat{a}^+\hat{a}\rangle)$

statistical properties of the coherent state

$$P_\alpha(n) = |\langle n|\alpha\rangle|^2 = \left|e^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}}\right|^2 = e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!}$$ (poisson as before mentioned)

average

$$\langle\hat{n}\rangle = \langle a^+a\rangle \;\hat{=}\; \langle\alpha|a^+a|\alpha\rangle = \sum_{n=0}^{\infty} n\cdot P_\alpha(n) = \ldots = |\alpha|^2$$ (MEAN Photon number)

$$\langle\hat{n}^2\rangle = \langle\alpha|\,a^+a\,a^+a\,|\alpha\rangle$$

$$= \langle\alpha|\,a^+(a^+a+1)a|\alpha\rangle = \langle\alpha|a^+a^+aa + a^+a|\alpha\rangle$$

$$= (|\alpha|^4 + |\alpha|^2)\langle\alpha|\alpha\rangle = |\alpha|^4 + |\alpha|^2$$

Therefore:

$$\langle\Delta\hat{n}^2\rangle \equiv \underset{\text{(VARIANCE)}}{(\langle\hat{n}^2\rangle - \langle\hat{n}\rangle)} = |\alpha|^4 = \langle\hat{n}\rangle^2$$

We thus call $|\alpha|^2$ MEAN PHOTON NUMBER

Coherence of coherent states: (cf. Scully Q.Optics Ch2)

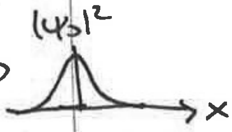
reminder: Coordinate representation $|x\rangle$ position coordinate (continuous basis)

it follows $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger)$ $\wedge$ $\hat{p} = \sqrt{2m\omega\hbar}\frac{(a - a^\dagger)}{i}$ $\left(= -i\hbar\nabla_x\right)$

or $\hat{a} = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{x} + \hbar\frac{\partial}{\partial x})$ operator representation.

$\underbrace{\hat{a}|\psi_0(x)\rangle}_{\text{groundstate } |0\rangle} = 0$ $\quad \langle x|a|0\rangle = \left(\omega \cdot x + \hbar\frac{\partial}{\partial x}\right)\psi_0(x) = 0$

$\Rightarrow$ $\psi_0(x) = \left(\frac{\omega}{\pi\hbar}\right)^{1/4} e^{-\omega x^2/2\hbar}$



Ground state $|0\rangle$ wave Function (minimum uncertainty state)

The coherent state is represented by:

$$\psi_0(x, t=0) = \left(\frac{\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{\omega}{2\hbar}(\underbrace{x - x_0}_{\text{displacement by } x_0})^2}$$

time evolution is given by (homework):

$$\boxed{\psi_0(x,t) = \left(\frac{\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{\omega}{2\hbar}(x - \underbrace{x_0\cos(\omega t)}_{\text{Harmonic oscillation around } x=0})^2}}$$

$\Rightarrow$ the coherent state does not decohere. It remains a minimum uncertainty state, i.e.:

$$\boxed{\Delta x \cdot \Delta p = \frac{\hbar}{2}}$$

where $\Delta x \equiv \left(\langle\hat{x}^2\rangle - \langle\hat{x}\rangle\right)^{1/2}$ and $\Delta p \equiv \left(\langle\hat{p}^2\rangle - \langle\hat{p}\rangle\right)^{1/2}$

with $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger)$ as above.

it follows that: $\langle\alpha|\hat{x}^2 - \hat{x}|\alpha\rangle\langle\alpha|\hat{p}^2 - \hat{p}|\alpha\rangle = \left(\frac{\hbar}{2}\right)^2$

$\rightarrow$ ie minimum uncertainty.

compare to vacuum $|0\rangle$:

$\langle 0|\hat{x}|0\rangle = 0$ $\quad \langle 0|\hat{x}^2|0\rangle = \frac{\hbar}{2m\omega}$

and $\langle 0|\hat{p}|0\rangle = 0$ $\quad \langle 0|\hat{p}^2|0\rangle = 2\hbar m\omega$

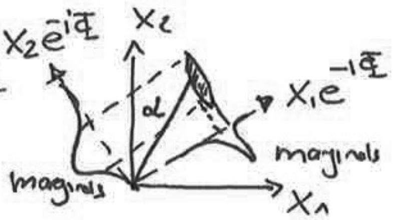
$\Bigg\}$ $\Delta x_0 \Delta p_0 = \frac{\hbar^2}{4}$

$\Rightarrow$ uncertainties in $\Delta x, \Delta p$ are identical to VACUUM STATE

$\Rightarrow$ A coherent state is a displaced vacuum state.

Quantum Optics I

Importantly; call $\boxed{|\alpha,\varepsilon\rangle = D(\alpha)S(\varepsilon)|0\rangle}$ SQUEEZED STATE



graphical representation of shifted and squeezed quadrature.

Properties (no proof):

$$\langle N\rangle = \langle \hat{a}^+\hat{a}\rangle = |\alpha|^2 + \sinh^2(r)$$

→ this implies for vacuum squeezed state $|0,\varepsilon\rangle$ that the photon (mean) number is non-zero (counter-intuitive!)

likewise:

$$\langle \Delta\hat{N}^2\rangle = \langle(\hat{a}^+\hat{a})^2\rangle = \langle\alpha,\varepsilon|(\hat{a}^+\hat{a})^2|\alpha,\varepsilon\rangle$$
$$= |\alpha\cosh(r) - \alpha^* e^{2i\Phi}\sinh(r)|^2 + 2\cosh^2 r\sinh^2 r$$

Hence again for squeezed vacuum the photon fluctuations are non-zero (counter intuitive).

$$\langle 0,\varepsilon|\Delta\hat{N}^2|0,\varepsilon\rangle = 2\cosh^2(r)\sinh^2(r)$$

TWO PHOTON COHERENT STATES

Consider $\hat{b} = \mu\hat{a} + \nu\hat{a}^+$ $\quad |\mu|^2 - |\nu|^2 = 1$

thus e.g. $\mu = \cosh(r)$ $\sinh(r) = \nu$

then $[\hat{b},\hat{b}^+] = 1$ and eigenstates of $\hat{b}$ are Two Photon Coherent States (Bogoliubov mode).

$$b|\beta\rangle_g = \beta\cdot\underbrace{|\beta\rangle_g}_{\text{eigenstate (Bogoliubov State)}}$$

The eigenstates are (no proof):

$$|\beta\rangle_g = \underbrace{S(\varepsilon)D(\alpha)}_{\substack{\text{squeezing operator}\\ \text{\& Displacement are reversed}}}|0\rangle \quad \text{and } \alpha \equiv \mu\cdot\beta - \beta^*\nu$$

Quantum Optics I

SQUEEZED STATES OF THE EM FIELD

Coherent states satisfy $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(a + a^+)$ ^ $\hat{p} = \sqrt{\frac{\hbar m\omega}{2}}\frac{(a - a^+)}{-i}$

variances: $(\Delta A)^2 \equiv \langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2$

which for coherent states $\Delta x^2 = \frac{\hbar}{2m\omega}$ ^ $\Delta p^2 = \frac{\hbar m\omega}{2}$ ^ $\Delta x^2 \Delta p^2 = \frac{\hbar^2}{4}$
(ie $\Delta x \Delta p = \frac{\hbar}{2}$)

we can define next quadratures $\hat{X}_1, \hat{X}_2$

$$\hat{a} = (\hat{X}_1 + i\hat{X}_2) \qquad \rightarrow \alpha = \langle \hat{X}_1 + i\hat{X}_2 \rangle$$

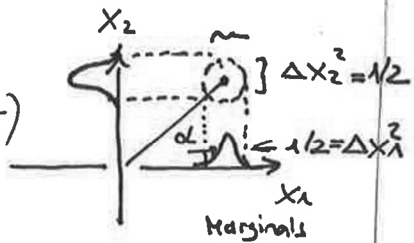
$$\hat{a}^+ = (\hat{X}_1 - i\hat{X}_2)$$

$\Longrightarrow$ Note: definition & normalization used in literature differ $1/2$ or $1/\sqrt{2}$ common, or 1

Commutator relation:

$$[\hat{X}_1, \hat{X}_2] = i \cdot \frac{1}{2}$$

and $\Delta X_1^2 = \Delta X_2^2 = \frac{1}{2}$ ie. $(\Delta X_1^2 \Delta X_2^2 = \frac{1}{4})$



Note that minimum uncertainty states ie $\Delta X_1^2 \Delta X_2^2 = \frac{1}{4}$ also exist with $\Delta X_1^2 < \frac{1}{2}$ $\Delta X_2^2 > \frac{1}{2}$
but $\Delta X_1^2 \Delta X_2^2 = \frac{1}{4}$

$\Rightarrow$ These are squeezed states of light.

Such states are generated via:

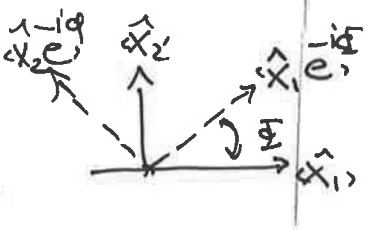
$$\underline{S(\varepsilon) = \exp(+\tfrac{1}{2}\varepsilon^* \hat{a}^2 - 1/2\, \varepsilon\, \hat{a}^{+2})}$$

(ie two photon processes eg parametric downconversion)

The operator obeys (no proof) $\qquad \varepsilon = re^{2i\varphi} \qquad$ (see next page proof)

$$S^+(\varepsilon)\, a\, S(\varepsilon) = \hat{a}\cosh(r) - \hat{a}^+ \sinh(r)\, e^{-2i\varphi}$$

$$S^+(\varepsilon)\, \underbrace{((\hat{X}_1 + i\hat{X}_2)e^{-i\Phi}}_{\text{rotated quadrature}} S(\varepsilon) = \hat{X}_1 e^{-i\Phi} e^{-r} + i\hat{X}_2 e^{-i\Phi} e^{+r}$$



Thus:

$\Delta X_1^2 = \frac{1}{2} e^{-2r}$	reduced below vacuum level
$\Delta X_2^2 = \frac{1}{2} e^{+2r}$	increased above

Lecture 2 -

Quantum Optics I

SQUEEZED STATES OF THE EM FIELD (Proof of operator transformation)

$$\boxed{\hat{S}(\xi) = e^{\left(+\frac{1}{2}\xi^* \hat{a}^2 + \frac{1}{2}\xi \hat{a}^{+2}\right)}}$$

Note that BAKER-HAUSDORFF:

$$\boxed{e^{\hat{B}}\hat{X}e^{-\hat{B}} = \hat{X} + [\hat{B},\hat{X}] + \frac{1}{2!}[\hat{B},[\hat{B},\hat{X}]] + \dots \frac{1}{n!}[\hat{B},[\hat{B}\dots[\hat{B},\hat{X}]\dots]]}$$

First Note that if $\hat{X} \equiv \hat{a}$

$$\hat{B} \equiv +\frac{1}{2}\xi^* \hat{a}^2 - \frac{1}{2}\xi \hat{a}^{+2}$$

Commutator ($\hat{a}\hat{a}^+ - \hat{a}^+\hat{a} = 1$)

$$[\hat{a},\hat{B}] \triangleq [\hat{a}, \frac{1}{2}\xi^*\hat{a}^2 - \frac{1}{2}\xi\hat{a}^{+2}] = [\hat{a}, -\frac{1}{2}\xi\hat{a}^{+2}] \quad \wedge [\hat{a},\hat{a}^+] = 1$$

$$= -\frac{1}{2}\xi(\hat{a}\hat{a}^{+2} - \hat{a}^{+2}a) = -\frac{1}{2}\xi(\underbrace{(\hat{a}^+a+1)}_{\triangleq aa^+}\hat{a}^+ - \hat{a}^+(a a^+ - 1))$$

$$= -\frac{1}{2}\xi \cdot 2\hat{a}^+ = \underline{-\xi\hat{a}^+} = -r\hat{a}^+ e^{i\varphi} \qquad \xi = re^{i\Theta}$$

Likewise:

$$[\hat{a}^+,\hat{B}] = [\hat{a}^+, \frac{1}{2}\xi^*\hat{a}^2 - \frac{1}{2}\xi(\hat{a}^+)^2] = -\frac{1}{2}\xi^*\hat{a} \times 2 = \underline{-\xi^*\hat{a} \triangleq -r\hat{a}e^{-i\varphi}}$$

Next

$$[\hat{B},[\hat{B},\hat{a}]] \qquad \frac{1}{3!}[\hat{B},[\hat{B},[\hat{B},a]]]$$

$$e^{-\hat{B}}\hat{a}e^{+\hat{B}} = \hat{a} + [\hat{B},\hat{a}] + \frac{1}{2!}[\dots] + \frac{1}{3!}[\dots] + \dots$$

$$= \hat{a} + (-\xi\hat{a}^+) + \frac{1}{2!}\underbrace{[\hat{B},(-\xi\hat{a}^+)]}_{\frac{1}{2!}(-\xi)[\xi^*\hat{a}]} + \frac{1}{3!}\underbrace{[\hat{B}[\hat{B},(-\xi\hat{a}^+)]]}_{\frac{(-\xi)}{3!}[\hat{B},-\xi\hat{a}^*]\ *} + \dots$$

$$\triangleq \frac{1}{3!}(\xi)^3 \cdot \hat{a}^+$$

$$\boxed{\dots = \hat{a}\left(1 + \frac{\xi^2}{2!} + \frac{\xi^4}{4!}\dots\right) + \hat{a}^+\left(+\xi + \frac{\xi^3}{3!} + \frac{\xi^5}{5!} + \dots\right)e^{-i\varphi}}$$

$$\underline{\hat{S}^+(r)\hat{a}\hat{S}(r) = \hat{a}.\cosh(r) + \hat{a}^+.\sinh(r)e^{i\varphi}} \qquad r \triangleq \xi$$

*Note that for even terms: $\xi \cdot \xi^* = r^2$ but odd $\xi \cdot \xi^* \xi = e^{-i\varphi}$

Likewise

$$\hat{S}(\xi)^+\hat{a}^+\hat{S}(\xi) = \hat{a}^+\cosh(r) - \hat{a}^*\sinh(r)e^{-i\varphi}$$ (no proof